Exercises on Catalan and Related Numbers

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by Richard P. Stanley

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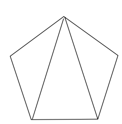
19. [1]–[3+] Show that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of elements of the 66 sets S_i, (a) $\leq i \leq$ (nnn) given below. We illustrate the elements of each S_i for $n = 3$, hoping that these illustrations will make any undefined terminology clear. (The terms used in (vv)–(yy) are defined in Chapter 7.) Ideally S_i and S_j should be proved to have the same cardinality by exhibiting a simple, elegant bijection $\phi_{ij} : S_i \to S_j$ (so 4290 bijections in all). In some cases the sets S_i and S_j will actually coincide, but their descriptions will differ.

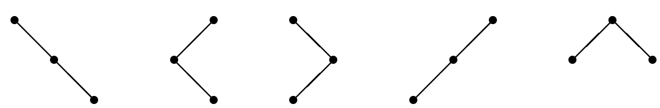
    (a) triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors

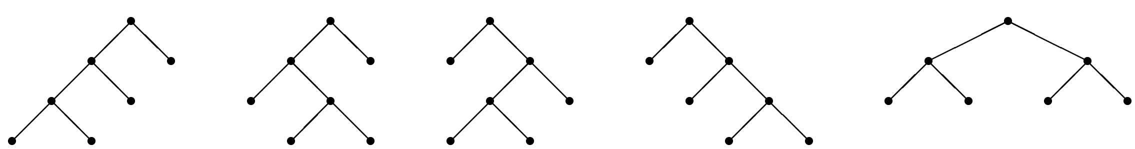
    (b) binary parenthesizations of a string of $n+1$ letters

    $$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

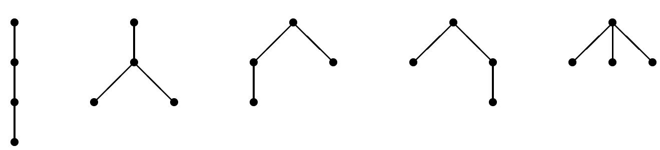
    (c) binary trees with n vertices

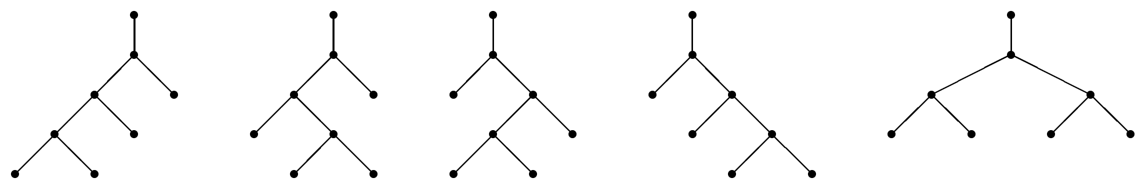
    (d) plane binary trees with $2n+1$ vertices (or $n+1$ endpoints)

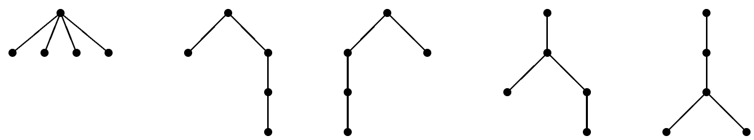
    (e) plane trees with $n+1$ vertices

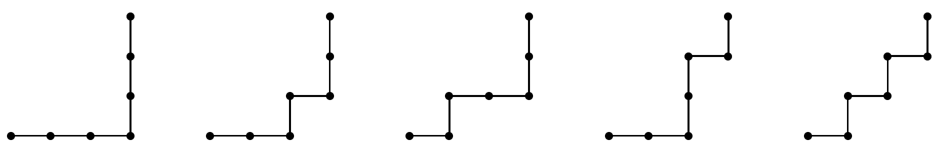
    (f) planted (i.e., root has degree one) trivalent plane trees with $2n+2$ vertices



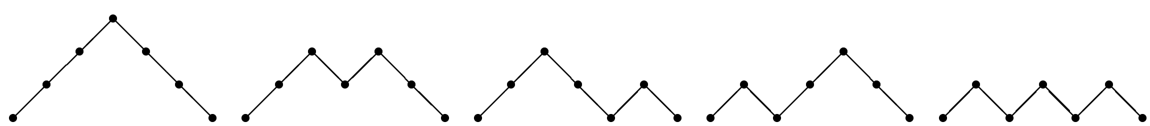
(g) plane trees with $n+2$ vertices such that the rightmost path of each subtree of the root has even length



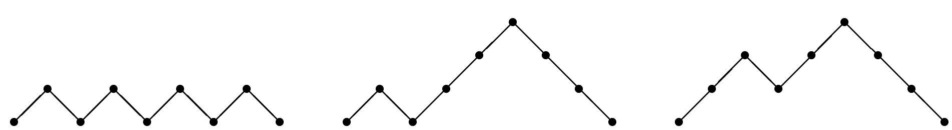
(h) lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y=x$

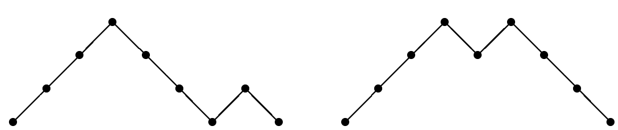


(i) Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis

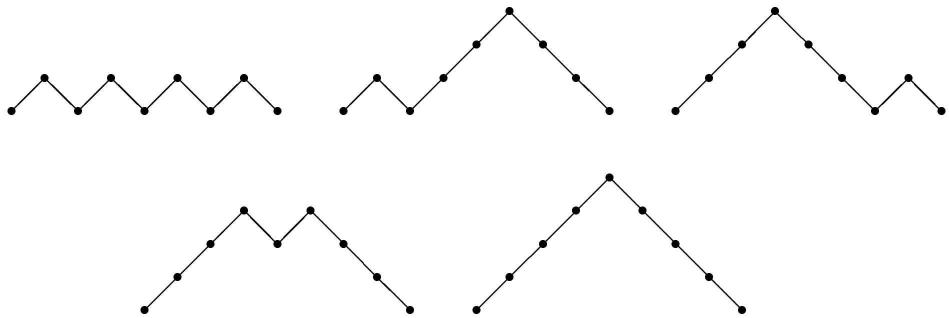


(j) Dyck paths (as defined in (i)) from $(0,0)$ to $(2n+2,0)$ such that any maximal sequence of consecutive steps $(1,-1)$ ending on the x-axis has odd length

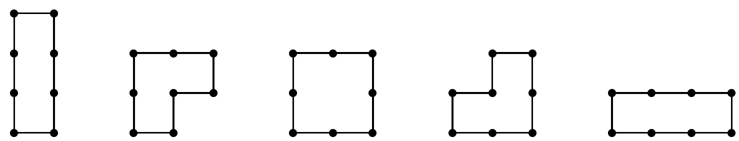




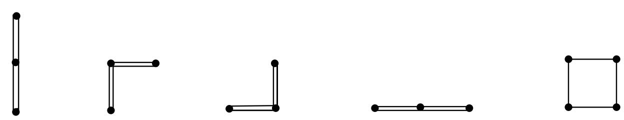
(k) Dyck paths (as defined in (i)) from $(0,0)$ to $(2n+2,0)$ with no peaks at height two.



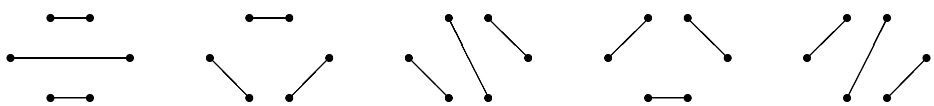
(l) (unordered) pairs of lattice paths with $n+1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, and only intersecting at the beginning and end



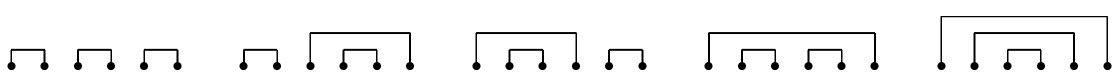
(m) (unordered) pairs of lattice paths with $n-1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, such that one path never arises above the other path



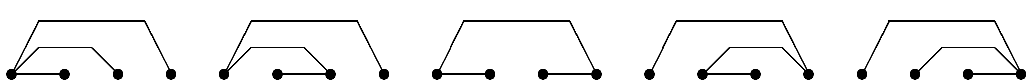
(n) n nonintersecting chords joining $2n$ points on the circumference of a circle



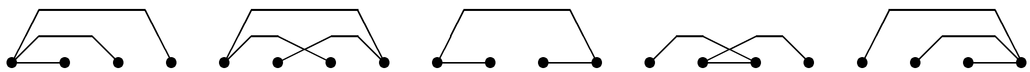
(o) ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points



(p) ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), and (δ) at every vertex, all the arcs exit in the same direction (left or right)



(q) ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no arc (including its endpoints) lies strictly below another arc, and (δ) at every vertex, all the arcs exit in the same direction (left or right)



(r) sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111{-}{-}{-} \qquad 11{-}1{-}{-} \qquad 11{-}{-}1{-} \qquad 1{-}11{-}{-} \qquad 1{-}1{-}1{-}$$

(s) sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$

$$111 \quad 112 \quad 113 \quad 122 \quad 123$$

(t) sequences $a_1 < a_2 < \cdots < a_{n-1}$ of integers satisfying $1 \leq a_i \leq 2i$

$$12 \quad 13 \quad 14 \quad 23 \quad 24$$

(u) sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$

$$000 \quad 001 \quad 010 \quad 011 \quad 012$$

(v) sequences $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \leq 1$ and all partial sums are nonnegative

$$0,0 \quad 0,1 \quad 1,-1 \quad 1,0 \quad 1,1$$

(w) sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_i \geq -1$, all partial sums are nonnegative, and $a_1 + a_2 + \cdots + a_n = 0$

$$0,0,0 \quad 0,1,-1 \quad 1,0,-1 \quad 1,-1,0 \quad 2,-1,-1$$

(x) sequences $a_1, a_2, \ldots, a_n$ of integers such that $0 \leq a_i \leq n-i$, and such that if $i < j$, $a_i > 0$, $a_j > 0$, and $a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0$, then $j - i > a_i - a_j$

$$000 \quad 010 \quad 100 \quad 200 \quad 110$$

(y) sequences $a_1, a_2, \ldots, a_n$ of integers such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$, then $a_j \leq a_i$

$$123 \quad 133 \quad 223 \quad 323 \quad 333$$

(z) sequences $a_1, a_2, \ldots, a_n$ of integers such that $1 \leq a_i \leq i$ and such that if $a_i = j$, then $a_{i-r} \leq j - r$ for $1 \leq r \leq j-1$

$$111 \quad 112 \quad 113 \quad 121 \quad 123$$

(aa) equivalence classes B of words in the alphabet $[n-1]$ such that any three consecutive letters of any word in B are distinct, under the equivalence relation $uijv \sim ujiv$ for any words u, v and any $i, j \in [n-1]$ satisfying $|i-j| \geq 2$

$$\{\emptyset\} \quad \{1\} \quad \{2\} \quad \{12\} \quad \{21\}$$

(For $n = 4$ a representative of each class is given by $\emptyset$, 1, 2, 3, 12, 21, 13, 23, 32, 123, 132, 213, 321, 2132.)

(bb) partitions $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ with $\lambda_1 \leq n-1$ (so the diagram of λ is contained in an $(n-1) \times (n-1)$ square), such that if $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ denotes the conjugate partition to λ then $\lambda'_i \geq \lambda_i$ whenever $\lambda_i \geq i$

$$(0,0) \quad (1,0) \quad (1,1) \quad (2,1) \quad (2,2)$$

(cc) permutations $a_1 a_2 \cdots a_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that: (i) the first occurrences of $1, 2, \ldots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$

112233 112332 122331 123321 122133

(dd) permutations $a_1 a_2 \cdots a_{2n}$ of the set $[2n]$ such that: (i) $1, 3, \ldots, 2n-1$ appear in increasing order, (ii) $2, 4, \ldots, 2n$ appear in increasing order, and (iii) $2i-1$ appears before $2i$, $1 \leq i \leq n$

123456 123546 132456 132546 135246

(ee) permutations $a_1 a_2 \cdots a_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called 321-*avoiding* permutations

123 213 132 312 231

(ff) permutations $a_1 a_2 \cdots a_n$ of $[n]$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-*avoiding* permutations)

123 132 213 231 321

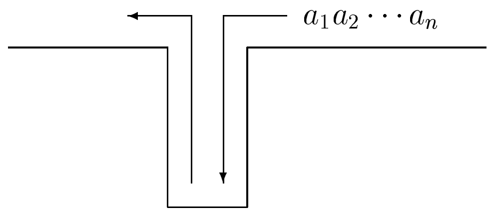
(gg) permutations w of $[2n]$ with n cycles of length two, such that the product $(1, 2, \ldots, 2n) \cdot w$ has $n+1$ cycles

$$\begin{aligned}
(1,2,3,4,5,6)(1,2)(3,4)(5,6) &= (1)(2,4,6)(3)(5) \\
(1,2,3,4,5,6)(1,2)(3,6)(4,5) &= (1)(2,6)(3,5)(4) \\
(1,2,3,4,5,6)(1,4)(2,3)(5,6) &= (1,3)(2)(4,6)(5) \\
(1,2,3,4,5,6)(1,6)(2,3)(4,5) &= (1,3,5)(2)(4)(6) \\
(1,2,3,4,5,6)(1,6)(2,5)(3,4) &= (1,5)(2,4)(3)(6)
\end{aligned}$$

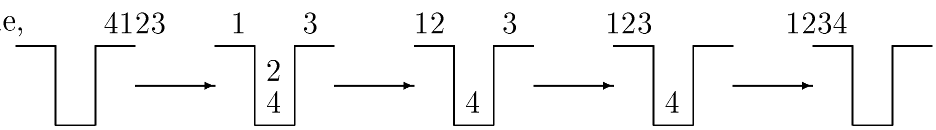
(hh) pairs (u, v) of permutations of $[n]$ such that u and v have a total of $n+1$ cycles, and $uv = (1, 2, \ldots, n)$

$$(1)(2)(3) \cdot (1,2,3) \qquad (1,2,3) \cdot (1)(2)(3) \qquad (1,2)(3) \cdot (1,3)(2)$$
$$(1,3)(2) \cdot (1)(2,3) \qquad (1)(2,3) \cdot (1,2)(3)$$

(ii) permutations $a_1 a_2 \cdots a_n$ of $[n]$ that can be put in increasing order on a single stack, defined recursively as follows: If $\emptyset$ is the empty sequence, then let $S(\emptyset) = \emptyset$. If $w = unv$ is a sequence of distinct integers with largest term n, then $S(w) = S(u)S(v)n$. A *stack-sortable* permutation w is one for which $S(w) = w$.

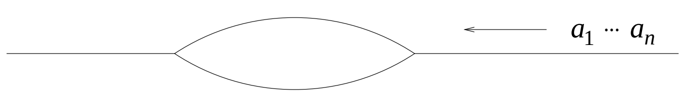


For example,



123 132 213 312 321

(jj) permutations $a_1 a_2 \cdots a_n$ of $[n]$ that can be put in increasing order on two parallel queues. Now the picture is



123 132 213 231 312

(kk) fixed-point free involutions w of $[2n]$ such that if $i < j < k < l$ and $w(i) = k$, then $w(j) \neq l$ (in other words, 3412-avoiding fixed-point free involutions)

$$(12)(34)(56) \quad (14)(23)(56) \quad (12)(36)(45) \quad (16)(23)(45) \quad (16)(25)(34)$$

(ll) cycles of length $2n+1$ in $\mathfrak{S}_{2n+1}$ with descent set $\{n\}$

2371456 2571346 3471256 3671245 5671234

(mm) Baxter permutations (as defined in Exercise 55) of $[2n]$ or of $[2n+1]$ that are reverse alternating (as defined at the end of Section 3.16) and whose inverses are reverse alternating

132546 153426 354612 561324 563412

1325476 1327564 1534276 1735462 1756342

(nn) permutations w of $[n]$ such that if w has ℓ inversions then for all pairs of sequences $(a_1, a_2, \ldots, a_\ell), (b_1, b_2, \ldots, b_\ell) \in [n-1]^\ell$ satisfying

$$w = s_{a_1} s_{a_2} \cdots s_{a_\ell} = s_{b_1} s_{b_2} \cdots s_{b_\ell},$$

where s_j is the adjacent transposition $(j, j+1)$, we have that the ℓ-element *multisets* $\{a_1, a_2, \ldots, a_\ell\}$ and $\{b_1, b_2, \ldots, b_\ell\}$ are equal (thus, for example, $w = 321$ is not counted, since $w = s_1 s_2 s_1 = s_2 s_1 s_2$, and the multisets $\{1, 2, 1\}$ and $\{2, 1, 2\}$ are not equal)

123 132 213 231 312

(oo) permutations w of $[n]$ with the following property: Suppose that w has ℓ inversions, and let

$$R(w) = \{(a_1, \ldots, a_\ell) \in [n-1]^\ell : w = s_{a_1} s_{a_2} \cdots s_{a_\ell}\},$$

where s_j is as in (nn). Then

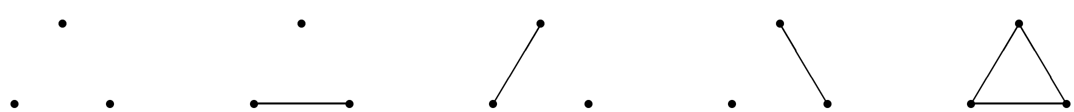
$$\sum_{(a_1,\ldots,a_\ell) \in R(w)} a_1 a_2 \cdots a_\ell = \ell!.$$

$$R(123) = \{\emptyset\}, \quad R(213) = \{(1)\}, \quad R(231) = \{(1,2)\}$$
$$R(312) = \{(2,1)\}, \quad R(321) = \{(1,2,1), (2,1,2)\}$$

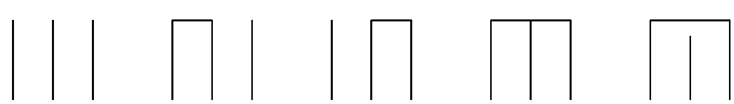
(pp) noncrossing partitions of $[n]$, i.e., partitions $\pi = \{B_1, \ldots, B_k\} \in \Pi_n$ such that if $a < b < c < d$ and $a, c \in B_i$ and $b, d \in B_j$, then $i = j$

123 12–3 13–2 23–1 1–2–3

(qq) partitions $\{B_1, \ldots, B_k\}$ of $[n]$ such that if the numbers $1, 2, \ldots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, \ldots, B_k$ are pairwise disjoint



(rr) noncrossing Murasaki diagrams with n vertical lines



(ss) noncrossing partitions of some set $[k]$ with $n+1$ blocks, such that any two elements of the same block differ by at least three

1–2–3–4 14–2–3–5 15–2–3–4 25–1–3–4 16–25–3–4

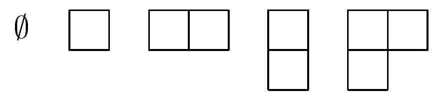
(tt) noncrossing partitions of $[2n+1]$ into $n+1$ blocks, such that no block contains two consecutive integers

137–46–2–5 1357–2–4–6 157–24–3–6 17–246–3–5 17–26–35–4 (uu) *nonnesting partitions* of $[n]$, i.e., partitions of $[n]$ such that if a, e appear in a block B and b, d appear in a *different* block B' where $a < b < d < e$, then there is a $c \in B$ satisfying $b < c < d$

123 12–3 13–2 23–1 1–2–3

(The unique partition of [4] that isn't nonnesting is 14–23.)

(vv) Young diagrams that fit in the shape $(n-1, n-2, \ldots, 1)$



(ww) standard Young tableaux of shape (n, n) (or equivalently, of shape $(n, n-1)$)

$$\begin{matrix}123\\456\end{matrix} \quad \begin{matrix}124\\356\end{matrix} \quad \begin{matrix}125\\346\end{matrix} \quad \begin{matrix}134\\256\end{matrix} \quad \begin{matrix}135\\246\end{matrix}$$

or

$$\begin{matrix}123\\45\end{matrix} \quad \begin{matrix}124\\35\end{matrix} \quad \begin{matrix}125\\34\end{matrix} \quad \begin{matrix}134\\25\end{matrix} \quad \begin{matrix}135\\24\end{matrix}$$

(xx) pairs (P, Q) of standard Young tableaux of the same shape, each with n squares and at most two rows

$$(123, 123) \quad \left(\begin{matrix}12\\3\end{matrix}, \begin{matrix}12\\3\end{matrix}\right) \quad \left(\begin{matrix}12\\3\end{matrix}, \begin{matrix}13\\2\end{matrix}\right) \quad \left(\begin{matrix}13\\2\end{matrix}, \begin{matrix}12\\3\end{matrix}\right) \quad \left(\begin{matrix}13\\2\end{matrix}, \begin{matrix}13\\2\end{matrix}\right)$$

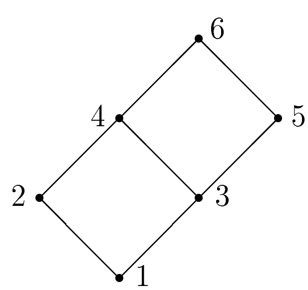
(yy) column-strict plane partitions of shape $(n-1, n-2, \ldots, 1)$, such that each entry in the ith row is equal to $n-i$ or $n-i+1$

$$\begin{matrix}3\,3\\2\end{matrix} \qquad \begin{matrix}3\,3\\1\end{matrix} \qquad \begin{matrix}3\,2\\2\end{matrix} \qquad \begin{matrix}3\,2\\1\end{matrix} \qquad \begin{matrix}2\,2\\1\end{matrix}$$

(zz) convex subsets S of the poset $\mathbb{Z} \times \mathbb{Z}$, up to translation by a diagonal vector (m, m), such that if $(i, j) \in S$ then $0 < i - j < n$.

$$\emptyset \quad \{(1,0)\} \quad \{(2,0)\} \quad \{(1,0),(2,0)\} \quad \{(2,0),(2,1)\}$$

(aaa) linear extensions of the poset $\mathbf{2} \times \mathbf{n}$



123456
123546
132456
132546
135246

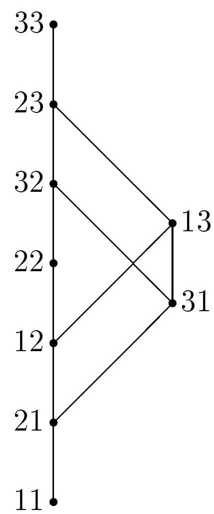
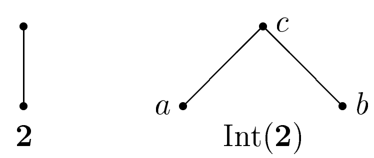


Figure 5: A poset with $C_4 = 14$ order ideals

(bbb) order ideals of $\mathrm{Int}(\boldsymbol{n-1})$, the poset of intervals of the chain $\mathbf{n}-\mathbf{1}$

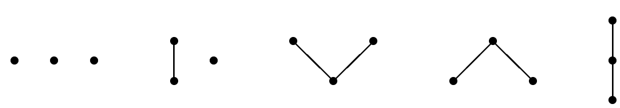


$$\emptyset, \quad a, \quad b, \quad ab, \quad abc$$

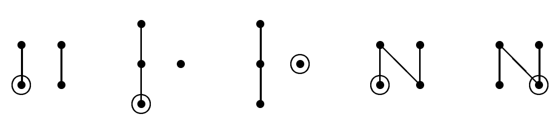
(ccc) order ideals of the poset A_n obtained from the poset $(\mathbf{n}-\mathbf{1})\times(\mathbf{n}-\mathbf{1})$ by adding the relations $(i,j)<(j,i)$ if $i>j$ (see Figure 5 for the Hasse diagram of A_4)

$$\emptyset \qquad \{11\} \qquad \{11,21\} \qquad \{11,21,12\} \qquad \{11,21,12,22\}$$

(ddd) nonisomorphic n-element posets with no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$



(eee) nonisomorphic $(n+1)$-element posets that are a union of two chains and that are not a (nontrivial) ordinal sum, rooted at a minimal element

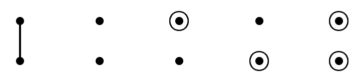


(fff) relations R on $[n]$ that are reflexive (iRi), symmetric ($iRj \Rightarrow jRi$), and such that if $1 \le i < j < k \le n$ and iRk, then iRj and jRk (in the example below we write

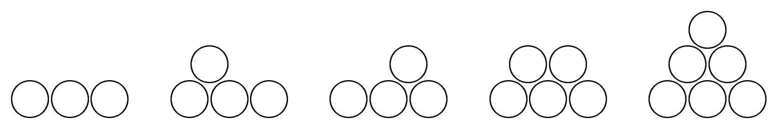
ij for the pair (i, j), and we omit the pairs ii)

$$\emptyset \quad \{12, 21\} \quad \{23, 32\} \quad \{12, 21, 23, 32\} \quad \{12, 21, 13, 31, 23, 32\}$$

(ggg) joining some of the vertices of a convex $(n-1)$-gon by disjoint line segments, and circling a subset of the remaining vertices



(hhh) ways to stack coins in the plane, the bottom row consisting of n consecutive coins



(iii) n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1 a_1 a_2 \cdots a_n 1$, each a_i divides the sum of its two neighbors

$$14321 \quad 13521 \quad 13231 \quad 12531 \quad 12341$$

(jjj) n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ whose elements sum to 0

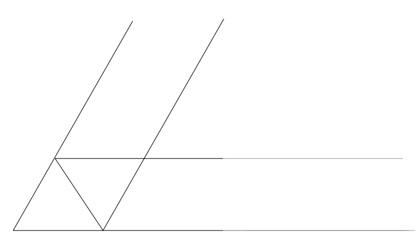
$$000 \quad 013 \quad 022 \quad 112 \quad 233$$

(kkk) n-element subsets S of $\mathbb{N} \times \mathbb{N}$ such that if $(i, j) \in S$ then $i \geq j$ and there is a lattice path from $(0, 0)$ to (i, j) with steps $(0, 1)$, $(1, 0)$, and $(1, 1)$ that lies entirely inside S

$$\{(0,0),(1,0),(2,0)\} \quad \{(0,0),(1,0),(1,1)\} \quad \{(0,0),(1,0),(2,1)\}$$

$$\{(0,0),(1,1),(2,1)\} \quad \{(0,0),(1,1),(2,2)\}$$

(lll) regions into which the cone $x_1 \geq x_2 \geq \cdots \geq x_n$ in $\mathbb{R}^n$ is divided by the hyperplanes $x_i - x_j = 1$, for $1 \leq i < j \leq n$ (the diagram below shows the situation for $n = 3$, intersected with the hyperplane $x_1 + x_2 + x_3 = 0$)



$$\begin{array}{ccccccccccccccccccccccccccc}
1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1\\
&1&&3&&2&&1&&5&&1&&2&&3&&1&&3&&2&&1&&5&\\
&&2&&5&&1&&4&&4&&1&&5&&2&&2&&5&&1&&4&&\\
&&&3&&2&&3&&3&&3&&2&&3&&3&&3&&2&&3&&&\\
&&&&1&&5&&2&&2&&5&&1&&4&&4&&1&&5&&&&\\
&&&&&2&&3&&1&&3&&2&&1&&5&&1&&2&&&&&\\
&&&&&&1&&1&&1&&1&&1&&1&&1&&1&&&&&&
\end{array}$$

Figure 6: The frieze pattern corresponding to the sequence $(1,3,2,1,5,1,2,3)$

(mmm) positive integer sequences $a_1, a_2, \ldots, a_{n+2}$ for which there exists an integer array (necessarily with $n+1$ rows)

$$\begin{array}{ccccccccccccccccccc}
1&&1&&1&&\cdots&&1&&1&&1&&\cdots&&1&&1\\
&a_1&&a_2&&a_3&&\cdots&&a_{n+2}&&a_1&&a_2&&\cdots&&a_{n-1}&\\
&&b_1&&b_2&&b_3&&\cdots&&b_{n+2}&&b_1&&\cdots&&b_{n-2}&&\\
&&&&&&&\vdots&&&&&&&&&&&\\
&&&r_1&&r_2&&r_3&&\cdots&&r_{n+2}&&r_1&&&&&\\
&&&&1&&1&&1&&\cdots&&1&&&&&&
\end{array} \tag{54}$$

such that any four neighboring entries in the configuration $\begin{smallmatrix} & r & \\ s & & t \\ & u & \end{smallmatrix}$ satisfy $st = ru+1$ (an example of such an array for $(a_1, \ldots, a_8) = (1,3,2,1,5,1,2,3)$ (necessarily unique) is given by Figure 6):

12213 22131 21312 13122 31221

(nnn) n-tuples $(a_1, \ldots a_n)$ of positive integers such that the tridiagonal matrix

$$\begin{bmatrix}
a_1 & 1 & 0 & 0 & \cdots & 0 & 0\\
1 & a_2 & 1 & 0 & \cdots & 0 & 0\\
0 & 1 & a_3 & 1 & \cdots & 0 & 0\\
 & & & & \vdots & & \\
0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1\\
0 & 0 & 0 & 0 & \cdots & 1 & a_n
\end{bmatrix}$$

is positive definite with determinant one

131 122 221 213 312

20. (a) [2+] Let m, n be integers satisfying $1 \leq n < m$. Show by a simple bijection that the number of lattice paths from $(1,0)$ to (m,n) with steps $(0,1)$ and $(1,0)$ that intersect the line $y = x$ in at least one point is equal to the number of lattice paths from $(0,1)$ to (m,n) with steps $(0,1)$ and $(1,0)$.

(b) [2–] Deduce that the number of lattice paths from $(0,0)$ to (m,n) with steps $(1,0)$ and $(0,1)$ that intersect the line $y=x$ only at $(0,0)$ is given by $\frac{m-n}{m+n}\binom{m+n}{n}$.

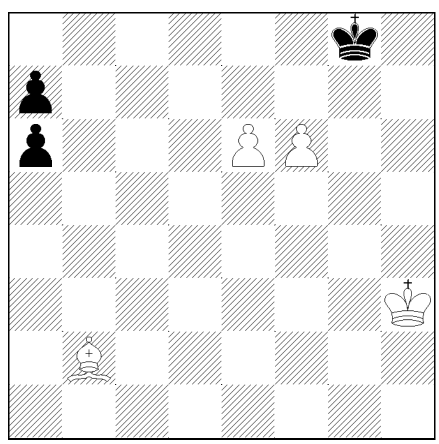
(c) [1+] Show from (b) that the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$ that never rise above the line $y=x$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. (This gives a direct combinatorial proof of interpretation (h) of C_n in Exercise 19.)

21. (a) [2+] Let X_n be the set of all $\binom{2n}{n}$ lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ and $(1,0)$. Define the *excedance* (also spelled "exceedance") of a path $P \in X_n$ to be the number of i such that at least one point (i,i') of P lies above the line $y=x$ (i.e., $i' > i$). Show that the number of paths in X_n with excedance j is independent of j.

(b) [1] Deduce that the number of $P \in X_n$ that never rise above the line $y=x$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (a direct proof of interpretation (h) of C_n in Exercise 19). Compare with Example 5.3.11, which also gives a direct combinatorial interpretation of C_n when written in the form $\frac{1}{n+1}\binom{2n}{n}$ (as well as in the form $\frac{1}{2n+1}\binom{2n+1}{n}$).

22. [2+] Show (bijectively if possible) that the number of lattice paths from $(0,0)$ to $(2n,2n)$ with steps $(1,0)$ and $(0,1)$ that avoid the points $(2i-1,2i-1)$, $1 \le i \le n$, is equal to the Catalan number C_{2n}.

23. [3–] Consider the following chess position.



Black is to make 19 consecutive moves, after which White checkmates Black in one move. Black may not move into check, and may not check White (except possibly on his last move). Black and White are *cooperating* to achieve the aim of checkmate. (In chess problem parlance, this problem is called a *serieshelpmate in 19*.) How many different solutions are there?